BAYES ADAPTIVE ESTIMATION OF THE POINT OF SHIFT

TO THE WEAR-OUT PHASE OF

RELIABILITY SYSTEMS

by

S. Zacks

TECHNICAL REPORT NO. 1

August 1, 1982

DEPARTMENT OF MATHEMATICAL SCIENCES
STATE UNIVERSITY OF NEW YORK
BINGHAMTON, NEW YORK

DTIC
SELECTED
AUG 0 5 1982
E

82 08 05 010

BAYES ADAPTIVE ESTIMATION OF THE
POINT OF SHIFT TO THE WEAR-OUT
PHASE OF RELIABILITY SYSTEMS

by

S. Zacks

State University of New York at Binghamton

Abstract

A new family of life distributions, called the wear-out distributions,
is developed on the basis of a failure rate function, which is a con-
stant up to the change-point and strictly increasing afterwards. Prop-
erties of these wear-out distributions are derived and a Bayes adaptive
procedures is developed for the estimation of the change point. Recur-
sive formulae are given for the determination of the posterior probability
that the change has occurred and of its Bayes estimator. The results of
numerical simulations are given to illustrate the properties of the adap-
tive procedure.

Key Words: Wear-out, Reliability System, Bayes Adaptive Procedure,
            Stopping Times, Failure Rate Functions, Simulations

1. Introduction

Consider a system having several components, whose life distributions depend on the state of the system. Each component is replaced immediately after failure (corrective maintenance), or at scheduled replacement epochs (preventive maintenance). The optimal scheduling of the replacement epochs depends on various economic considerations, and on the particular life distribution of the component. This in turn depends on the state of the whole system. We recognize two phases of the system. Phase I is the phase in which the components have a constant failure rate, λ. Phase II is the phase in which the failure rate is greater than λ and increasing. Phase I is a mature stable phase of the system, while Phase II is a wear-out phase. It is important to detect early the transition of the system from Phase I to Phase II. Optimal replacement scheduling and inventory management during the wear-out phase of the system are different from those appropriate to the constant failure rate phase. The present paper is devoted to the study of the detection of the shift from Phase I to Phase II, and estimating the change point, τ, where this shift occurs.

In Section 2 we derive from a specified failure rate model an appropriate life distribution for the present study. This life distribution is a composition of an exponential distribution with a Weibull. It seems to be new in the sense of not being dealt with in the literature on reliability. We shall call it the wear-out life distribution. Several properties of this distribution are presented in Section 2. In Section 3 we develop a Bayes adaptive procedure of estimating the change-point τ, on the basis of a sequence of the replacement times of the component. Section 4 is devoted to the derivation of some recursive formulae, which are required by the adaptive procedure. In Section 5 we provide some simulation results, which provide information of the characteristics of the detection procedure.

2. The Wear-Out Life Distribution

The wear-out life distribution is based on the following non-decreasing failure-rate function

$$h(t;\lambda,\alpha,\tau) = \begin{cases} \lambda & \text{if } 0 \leq t \leq \tau_+ \\[2ex] \lambda + \lambda^\alpha \alpha (t-\tau_+)^{\alpha-1} & \text{, if } \tau < t \end{cases} \qquad (2.1)$$

where λ is the constant failure-rate in Phase I, $0 < \lambda < \infty$; α is a shape parameter, $1 \leq \alpha < \infty$ and τ is the change point parameter $\tau_+ = \max(0,\tau)$. If $\tau \leq 0$ the system is already in Phase II. The present model represents a superposition of a Weibull failure-rate function (see Mann, Schafer and Singpurwalla [1;pp.127]) on a constant failure-rate function for $t \geq \tau_+$. Notice also that the limiting case of $\alpha = 1$ corresponds to a shift at τ from an exponential life distribution with MTBF $\mu_1 = 1/\lambda$ to an exponential life distribution with an MTBF $\mu_2 = 1/2\lambda$, i.e., for all $t > \tau$, $p\{X \geq t \mid x \geq \tau\} = \exp\{-2\lambda(t-\tau)\}$. From this failure-rate model we obtain a life distribution, called the wear-out life distribution, with a c.d.f.

$$F(x;\lambda,\alpha,\tau) = 1-\exp\{-\lambda \int_0^x (1+\alpha(\lambda(t-\tau_+)^{\alpha-1})dt\}$$

$$= 1-\exp\{-\lambda x - (\lambda(x-\tau_+)_+)^\alpha\} \ , \ x \geq 0 \qquad (2.2)$$

Obviously, $F(x;\lambda,\alpha,\tau) = 0$ for $x \leq 0$.
The corresponding p.d.f. is

$$f(x;\lambda,\tau,\alpha) = \lambda e^{-\lambda x}(1+\alpha(\lambda(x-\tau_+)_+)^{\alpha-1}) \ .$$

$$\cdot \exp\{-\lambda^\alpha(x-\tau_+)_+^\alpha\} \ , \ x \geq 0 \ . \qquad (2.3)$$

It follows the conditional p.d.f. of X (the life length), given $\{X \geq \tau\}$, which is the p.d.f. of the residual life distribution in Phase II, is

$$g(x-\tau_+;\lambda,\alpha) = \lambda e^{-\lambda(x-\tau+)}(1+\alpha(\lambda(x-\tau_+))^{\alpha-1}) \ .$$

$$\cdot \exp\{-\lambda^\alpha(x-\tau_+)^\alpha\} \ , \ x \geq \tau \qquad (2.4)$$

Notice that $\beta = 1/\lambda$ is a scale parameter of this distribution, and that τ_+ is a location parameter of the truncation type (see Zacks [3;pp.14d]).
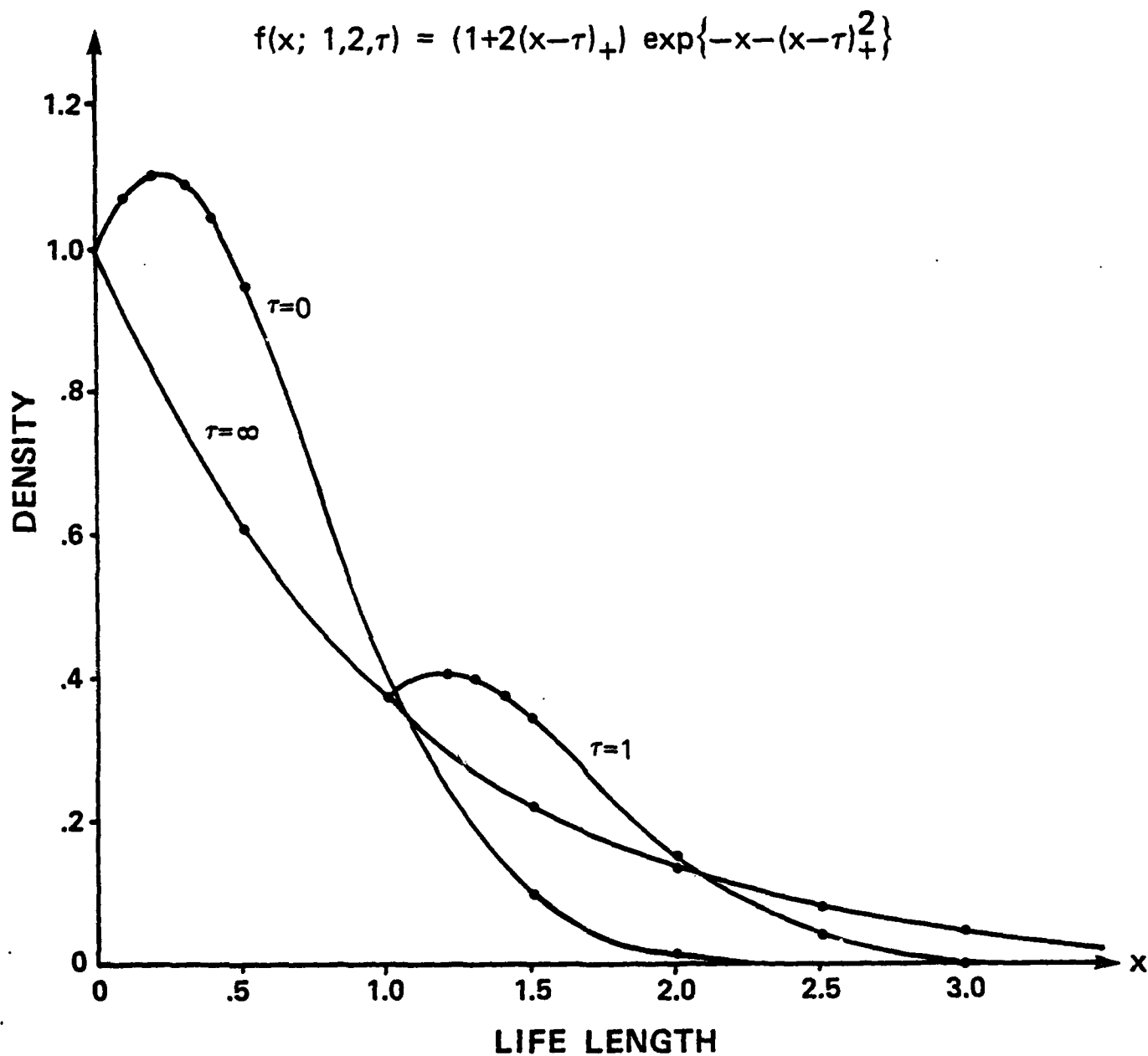
$$f(x; 1,2,\tau) = (1+2(x-\tau)_+) \exp\{-x-(x-\tau)^2_+\}$$

Figure 1. Density functions of the wear-out distribution
for $\lambda=1$, $\alpha=2$ and $\tau=0$, 1 and ∞ (exponential).

In Fig. 1 we present graphs of the wear-out life density (2.3) for $\alpha=2$, $\lambda=1$ and various values of τ .   We remark here that, if τ is very large compared to $1/\lambda$, the wear-out distribution is practically an exponential distribution with mean $\mu=1/\lambda$. This is the case when the system is far from its wear-out phase. We conclude the present section with a presentation of the formula for the moments of the wear-out distribution (2.3). Since $1/\lambda$ is a scale parameter, we will present the moment $\mu_r(\alpha,\tau)$ for the case of $\lambda=1$. If $\lambda\neq1$ then

$$E\{X^r;\lambda,\alpha,\tau\} = \frac{1}{\lambda^r} \mu_r(\alpha,\lambda\tau) .$$

The r-th moment in the standard case is for $\tau>0$,

$$\mu_r(\alpha,\tau) = \int_0^\tau x^r e^{-x}dx + \int_\tau^\infty x^r e^{-x-(x-\tau)^\alpha}(1+\alpha(x-\tau)^{\alpha-1})dx$$

$$= r!(1-Pos(r|\tau))$$

$$+ e^{-\tau} \sum_{j=0}^r \binom{r}{j}\tau^{r-j}M_j(\alpha) \qquad , \tag{2.5}$$

where

$$Pos(k|\lambda) = e^{-\lambda} \sum_{j=0}^k \frac{\lambda^j}{j!} \tag{2.6}$$

is the Poisson c.d.f. and

$$M_j(\alpha) = \int_0^\infty y^j e^{-y-y^\alpha}(1+\alpha\cdot y^{\alpha-1})dy \qquad , j\geq0 \tag{2.7}$$

Integration by parts yields, for every $j\geq1$,

$$M_j(\alpha) = j \int_0^\infty y^{j-1}e^{-(y+y^\alpha)}dy \qquad , \tag{2.8}$$

which is smaller than $\Gamma(j+1) = j!$ for all $j\geq1$. Notice that $M_0(\alpha)=1$ for all $\alpha\geq1$. The integral in (2.8) can be evaluated numerically. In the case of $\alpha=2$ we obtain, for all $j\geq1$,

$$M_j(2) = je^{1/4} \int_0^\infty y^{j-1}e^{-(y+1/2)^2}dy \tag{2.9}$$

$$= \frac{je^{1/4}}{\sqrt{2}} \sum_{\ell=0}^{j-1} (-1)^\ell \binom{j-1}{\ell}(\frac{1}{2})^{j-1-\ell/2}B(\ell;\sqrt{1/2}) \qquad ,$$

where, for all $\ell \geq 2$,

$$B(\ell;\xi) = \int_{\xi}^{\infty} z^{\ell} e^{-\frac{1}{2}z^2} dz \qquad (2.10)$$

$$= \xi^{\ell-1} \exp\{-\frac{1}{2}\xi^2\} + (\ell-1) B(\ell-2;\xi)$$

and

$$B(0;\xi) = \sqrt{2\pi} (1-\Phi(\xi))$$

$$(2.11)$$

$$B(1;\xi) = \exp\{-\frac{1}{2}\xi^2\} \quad ,$$

where $\Phi(z)$ is the standard normal c.d.f.

The functions $B(\ell; \frac{1}{2})$ can be determined recursively, according to (2.10) and (2.11). In Table 2.1 we provide the first four moments, the standard deviation σ , the measures of skewness γ_1 and kurtosis, γ_2 , of the wear-out distribution for $\lambda=1$, $\alpha=2$ and several values of τ .

Table 2.1 The first four moments, standard deviation, and measure of skewness and kurtosis of the wear-out distribution, for $\lambda=1$, $\alpha=2$ and $\tau=0(.5)2$, $3(1)7$.

τ	0	.5	1.0	1.5	2.0
μ_1	0.54564	0.72442	0.83285	0.89862	0.93851
μ_2	0.45436	0.78694	1.09709	1.35098	1.54486
μ_3	0.47769	1.03764	1.76118	2.53153	3.25967
μ_4	0.59026	1.55837	3.18161	5.34890	7.86491
σ	0.39577	0.51201	0.63518	0.73720	0.81490
γ_1	0.94918	0.65373	0.68460	0.85060	1.04106
γ_2	3.80641	2.95378	2.68424	2.87374	3.32183

τ	3	4	5	6	7
μ_1	0.97738	0.99168	0.99694	0.99887	0.99959
μ_2	1.78732	1.90511	1.95897	1.98265	1.99279
μ_3	4.42171	5.15969	5.57697	5.79571	5.90446
μ_4	12.93668	17.10481	20.00519	21.81282	22.85481
σ	0.91217	0.96005	0.98239	0.99242	0.99680
γ_1	1.38130	1.63008	1.79281	1.89034	1.94467
γ_2	4.55947	5.85946	6.96183	7.77159	8.30424

The exponential distribution is the limiting case when $\tau \to \infty$. The
moments of the exponential distribution with $\lambda = 1$ are $\mu_1 = 1$, $\mu_2 = 2$,
$\mu_3 = 6$ and $\mu_4 = 24$. We see in Table 2.1 that when $\tau = 7$ the moments
of the wear-out distribution are already close to those of the limiting
exponential distribution.

3. Bayes Estimation of the Change Point

In the present section we develop the Bayesian framework for estimating the change point, τ , and deciding whether the transition to Phase II has taken place.

Given a sequence $t_1, t_2, \ldots$ of replacement epochs, where $0 < t_1 < t_2 < \ldots$, we consider the random variables $X_i = t_i - t_{i-1} (i=1,2,\ldots)$, where $t_0 \equiv 0$. X_i represents the operational time length of i-th sequentially installed component. Since the preventive maintenance replaces the component after Δ units of time, $X_i = \min(Y_i, \Delta)$, where $Y_1, Y_2, \ldots$ are independent random variables having the wear-out distribution, with parameters λ, α and $\tau_i = \tau - t_{i-1}$ $(i=1,2,\ldots)$. τ_i is the length of time left for the system, after the i-th installment of the component till the transition to Phase II. If $\tau_i < 0$, the chang to Phase II has taken place before t_{i-1} .

In the present study we assume that λ and α are known. The only unknown parameter is the change point, τ . Without loss of generality we assume that $\lambda=1$. If $\lambda \neq 1$ replace X_i in the formulae below by $\lambda X_i (i=1,\ldots,n)$.

Given n replacement points $t_1 < t_2 < \ldots < t_n$, let $\underset{\sim}{X}^{(n)} = (X_1, \ldots, X_n)'$ where $X_i = t_i - t_{i-1}$, and define

$$
J_i = \begin{cases} 1 & , \text{ if } X_i < \Delta \\ 0 & , \text{ if } X_i = \Delta \end{cases} \tag{3.1}
$$

The likelihood function of τ , given $\underset{\sim}{X}^{(n)}$, is

$$
L(\tau; \underset{\sim}{X}^{(n)}) = I\{\tau \leq 0\} \exp\{-\sum_{i=1}^{n} X_i^{\alpha}\} \prod_{i=1}^{n} (1+\alpha J_i X_i^{\alpha-1})
$$

$$
+ \sum_{j=1}^{n-1} I\{t_{j-1} < \tau \leq t_j\} \exp\{-\sum_{k=j+1}^{n} X_k^{\alpha}\} \prod_{k=j+1}^{n} (1+\alpha J_k X_k^{\alpha-1}) \; .
$$

$$
[1+\alpha J_j (X_j + t_{j-1} - \tau)^{\alpha-1}] \exp\{-(X_j + t_{j-1} - \tau)^{\alpha}\}
$$

$$
+ I\{t_{n-1} < \tau \leq t_n\} [1+\alpha J_n (X_n + t_{n-1} - \tau)^{\alpha-1}] \exp\{-(X_n + t_{n-1} - \tau)^{\alpha}\}
$$

$$
+ I\{\tau > t_n\} \qquad , \tag{3.2}
$$

where $I\{A\}$ is the indicator function of the set A.

Generally, let $\xi(\tau)$ denote the prior p.d.f. of τ . Then, the posterior p.d.f. of τ , given $\underset{\sim}{X}^{(n)}$, is

$$\xi(\tau|\underset{\sim}{X}^{(n)}) = \xi(\tau)L(\tau;\underset{\sim}{X}^{(n)})/D(\xi;\underset{\sim}{X}^{(n)}) \quad , \tag{3.3}$$

where

$$D(\xi;\underset{\sim}{X}^{(n)}) = \int L(t;\underset{\sim}{X}^{(n)})dP_\xi(\tau \leq t) \quad . \tag{3.4}$$

In the present study we consider a prior distribution with p.d.f.

$$\xi(\tau) = \begin{cases} p & , \text{ if } \tau \leq \tau_0 \\ (1-p)\psi e^{-\psi(\tau-\tau_0)} & , \text{ if } \tau > \tau_0 \quad , \end{cases} \tag{3.5}$$

where $0<p<1$ and $0<\psi<\infty$. τ_0 is a time point such that, with high prior probability, $1-p$, the true wear-out point, τ , exceeds it. We are interested therefore in time points greater than τ_0 . We shall assume accordingly, without loss of generality, that $\tau_0=0$. For the prior p.d.f. (3.5) under consideration, the function (3.4) assumes the form

$$\begin{aligned} D(p,\psi;\underset{\sim}{X}^{(n)}) &= p \exp\{-\sum_{i=1}^{n} X_i^\alpha\} \prod_{i=1}^{n} [1+\alpha J_i X_i^{\alpha-1}] \\ &+ (1-p) \sum_{j=1}^{n-1} \exp\{-\sum_{k=j+1}^{n} X_k^\alpha\} \prod_{k=j+1}^{n} [1+\alpha J_k X_k^{\alpha-1}] \quad . \\ &\cdot e^{-\psi t_{j-1}} \psi \int_0^{X_j} e^{-\psi u} [1+\alpha J_j (X_j-u)^{\alpha-1}] \; e^{-(X_j-u)^\alpha} du \\ &+ (1-p)\psi e^{-\psi t_{n-1}} \int_0^{X_n} e^{-\psi u} [1+\alpha J_n (X_n-u)^{\alpha-1}] \; e^{-(X_n-u)^\alpha} du \\ &+ (1-p) e^{-\psi t_n} \quad . \end{aligned} \tag{3.6}$$

The posterior probability, after observing $\underset{\sim}{X}^{(n)}$, that the transition to Phase II has already taken place (i.e., $\{\tau \leq t_n\}$) is $p_n = 1-q_n$, where

$$q_n = (1-p)e^{-\psi t_n}/D(p,\psi;\underset{\sim}{X}^{(n)}) \quad . \tag{3.7}$$

If p_n is large we have high evidence that the change has already taken place. In addition, after observing $\underset{\sim}{X}^{(n)}$ the Bayes estimator of τ , for squared error loss function, is the posterior expectation of τ , given $\underset{\sim}{X}^{(n)}$. This estimator is determined by the formula

$$\hat{\tau}_n(p,\psi;\underset{\sim}{X}^{(n)}) = E\{\tau|\underset{\sim}{X}^{(n)},p,\psi\} =$$

$$\{(1-p)\psi \sum_{j=1}^{n-1} \prod_{k=j+1}^{n} [1+\alpha J_k X_k^{\alpha-1}] \exp\{-\sum_{k=j+1}^{n} X_k^{\alpha}\} \quad .$$

$$\cdot e^{-\psi t_{j-1}} \int_0^{X_j} (t_{j-1}+u)e^{-\psi u} [1+\alpha J_j(X_j-u)^{\alpha-1}] e^{-(X_j-u)^{\alpha}} du$$

$$+ (1-p)\psi e^{-\psi t_{n-1}} \int_0^{X_n} (t_{n-1}+u)e^{-\psi u} [1+\alpha J_n(X_n-u)^{\alpha-1}] e^{-(X_n-u)^{\alpha}} du$$

$$+ (1-p)\psi e^{-\psi t_n} \int_0^{\infty} (t_n+u)e^{-\psi u}du\} \div D(p,\psi;\underset{\sim}{X}^{(n)}) \quad . \qquad (3.8)$$

Notice that if $\lambda\neq1$ we determine p_n and $\hat{\tau}_n(p,\psi;X^{(n)})$ by substituting in the above formula λX_i for X_i and ψ/λ for ψ .

Thus, we can compute p_n and $\hat{\tau}_n$ adaptively, after each replacement, and decide that the change has occurred at the first time $\hat{\tau}_n \leq t_n$ or p_n is sufficiently large. Shiryaev [2] studied the problem of the quickest detection of the occurrence of a change in a specifided distribution. He showed that, when the distributions before and after the change are known, *the optimal Bayes sequential stopping time is at the smallest* $n\geq1$ *for which* $p_n \geq \Pi^*$. Π^* is a threshold that can be determined to minimize the total expected loss. Our problem is however different. First, we do not have just two known distributions with a switch from one to another at an unknown change-point, τ , but a sequence of wear-out distributions with unknown parameters τ_i , and we wish to detect the time point at which, for the first time, $\bar{\tau}_i \leq 0$. Moreover, we do not have a sequential stopping problem, but an adaptive decision problem. The replacement process is continuing also after the decision that $\tau_i \leq 0$ has been reached. The implications of this decision are on the adaptation of the scheduled replacement period, Δ , and on the inventory of spare components. In the following section we provide some recursive formulae for the determination of $D(p,\psi;\underset{\sim}{X}^{(n)})$, p_n and $\hat{\tau}_n(p,\psi;\underset{\sim}{X}_n)$.

4. Recursive Formulae

From (3.3) and (3.5) one immediately obtains that, for every $t \geq t_n$ $(n=1,2,\ldots)$

$$P\{\tau \geq t \mid \underset{\sim}{X}^{(n)}\} = \frac{(1-p)}{D(p,\psi;\underset{\sim}{X}^{(n)})} e^{-\psi t} \qquad (4.1)$$

Combining this result with (3.7) we obtain for all $t \geq t_n$

$$P\{\tau \geq t \mid \underset{\sim}{X}^{(n)}\} = (1-p_n)e^{-\psi(t-t_n)} \quad , \qquad (4.2)$$

where p_n is the posterior probability of $\{\tau \leq t_n\}$. This yields that, after observing $\underset{\sim}{X}^{(n)}$, the posterior distribution of τ for $\tau \geq t_n$ is of the same form as (3.1), with p replaced by p_n , i.e., the posterior p.d.f. of τ , given $\underset{\sim}{X}^{(n)}$ satisfies

$$\xi_n(\tau) = \begin{cases} p_n & , \text{ if } \tau \leq t_n \\ (1-p_n)\psi e^{-\psi(\tau-t_n)} & , \text{ if } \tau > t_n \end{cases} \qquad (4.3)$$

According to (3.6), the function $D_n \equiv D(p,\psi;\underset{\sim}{X}^{(n)})$, $n \geq 1$, can be computed by the following recursive formula

$$D_n = D_{n-1}(1+\alpha J_n X_n^{\alpha-1}) \exp\{-X_n^\alpha\} + (1-p)e^{-\psi t_{n-1}}$$
$$\cdot [\psi \int_0^{X_n} e^{-\psi u}(1+\alpha J_n(X_n-u)^{\alpha-1}) \exp\{-(X_n-u)^\alpha\}du$$
$$- (1+\alpha J_n X_n^{\alpha-1}) \exp\{-X_n^\alpha\}] + (1-p)e^{-\psi t_n} \quad , \qquad (4.4)$$

where $D_0 \equiv 1$.

Notice that for all λ , $0 < \lambda < \infty$,

$$D(p,\psi;\underset{\sim}{X}^{(n)}) = D(p,\psi/\lambda;\lambda\underset{\sim}{X}^{(n)}) \quad . \qquad (4.5)$$

Employing (3.7) and (4.4) we obtain

$$1-p_n = \frac{(1-p)e^{-\psi t_n}}{D_n} = \frac{(1-p)e^{-\psi t_{n-1}}}{D_{n-1}} \cdot \frac{D_{n-1}}{D_n} e^{-\psi X_n}$$
$$= (1-p_{n-1}) \frac{e^{-\psi X_n}}{D_n/D_{n-1}} \quad , \qquad n \geq 1 \quad . \qquad (4.6)$$

Moreover,

$$\frac{D_n}{D_{n-1}} = p_{n-1}(1+\alpha J_n X_n^{k-1}) \exp\{-X_n^\alpha\} +$$

$$(1-p_{n-1})\psi \int_0^{X_n} e^{-\psi u}(1+\alpha J_n(X_n-u)^{\alpha-1}) \exp\{-(X_n-u)^\alpha\}du$$

$$+ (1-p_{n-1})e^{-\psi X_n} \tag{4.7}$$

Thus, define for all $0 \le x \le \Delta$ the function

$$\Psi_\alpha(p,\psi,x) = pe^{-\psi x}(1+\alpha J x^{\alpha-1}) +$$

$$(1-p)\psi \int_0^x e^{-\psi u}(1+\alpha J(x-u)^{\alpha-1}) \exp\{-(x-u)^\alpha\}du$$

$$+ (1-p) \exp\{-\psi x\} \quad , \tag{4.8}$$

where $J = I\{x < \Delta\}$. It follows from (4.7) and (4.8) that, for every $n \ge 1$

$$D(p,\psi;\underset{\sim}{X}^{(n)}) = D(p,\psi;\underset{\sim}{X}^{(n-1)}) \ \Psi_\alpha(p_{n-1},\psi,X_n) \quad , \tag{4.9}$$

and

$$p_n = 1 - \frac{(1-p_{n-1}) \exp\{-\psi X_n\}}{\Psi_\alpha(p_{n-1},\psi,X_n)} \tag{4.10}$$

The integral in (4.8) can be evaluated numerically, or according to the formula

$$(1-p)\psi \int_0^x e^{-\psi u}(1+\alpha J(x-u)^{\alpha-1})e^{-(x-u)^\alpha}du =$$

$$(1-p) \frac{\psi}{\alpha} e^{-\psi x} \sum_{j=0}^\infty \frac{\psi^j}{j!} [\Gamma(x^\alpha; \frac{j+1}{\alpha}) + \alpha J\Gamma(x^\alpha; \frac{j}{\alpha} + 1)] \quad , \tag{4.11}$$

where $\Gamma(x;\nu) = \int_0^x z^{\nu-1}e^{-z}dz$ is the incomplete gamma function $(\nu > 0)$.

In the case of $\alpha = 2$ we obtain the explicit formula

$$\Psi_2(p,\psi,x) = pe^{-x^2}(1+2Jx) +$$

$$(1-p)\psi e^{-\psi x+\psi^2/4} \sqrt{\pi} \; [\phi(\sqrt{2}(x - \frac{\psi}{2})) + \Phi(\frac{\psi}{\sqrt{2}}) -1] \; (1+J\psi)$$

$$+J(1-p)\psi e^{-\psi x+\psi^2/4} \sqrt{2\pi} \; [\phi(\frac{\psi}{\sqrt{2}}) - \phi(\frac{\psi}{\sqrt{2}} - \sqrt{2}x)]$$

$$+(1-p)e^{-\psi x} \qquad , \tag{4.12}$$

where $\phi(z)$ and $\Phi(z)$ are, respectively, the p.d.f. and the c.d.f. of the standard normal distribution.

We discuss now the recursive determination of the Bayes estimator of τ. According to (3.8)

$$\hat{\tau}_n(p,\psi;\underset{\sim}{X}^{(n)}) = E_n + (1-p_n)\,(t_n + \frac{1}{\psi}) \qquad , \tag{4.13}$$

where

$$E_n = \Big\{ (1-p)\psi \sum_{j=1}^{n-1} \prod_{k=j+1}^{n} [1+\alpha J_k X_k^{\alpha-1}] \exp\{-\sum_{k=j+1}^{n} X_k^{\alpha}\}$$

$$e^{-\psi t_{j-1}} \int_0^{X_j} (t_{j-1}+u) e^{-\psi u}[1+\alpha J_j(X_j-u)^{\alpha-1}] \exp\{-(X_j-u)^{\alpha}\}du$$

$$+ (1-p)\psi e^{-\psi t_{n-1}} \int_0^{X_n} (t_{n-1}+u) e^{-\psi u} [1+\alpha J_n(X_n-u)^{\alpha-1}] \exp\{-(X_n-u)^{\alpha}\}du \Big\} \div D_n$$

$$n \geq 2 \tag{4.14}$$

Define the function

$$\eta_\alpha(p,\psi,t,x) = (1-p)e^{-\psi t}\psi \int_0^x (t-y)\,(1+\alpha Jy^{\alpha-1})e^{\psi y-y^\alpha}dy \qquad . \tag{4.15}$$

Then,

$$E_1 = \eta_\alpha(p,\psi,t_1,X_1)/D_1 \qquad , \tag{4.16}$$

and, for every $n \geq 1$,

$$E_{n+1} = [E_n D_n(1+\alpha J_{n+1}X_{n+1}^{\alpha-1}) \exp\{-X_{n+1}^{\alpha}\}$$

$$+ \eta_\alpha(p,\psi,t_{n+1},X_{n+1})] \div D_{n+1} \qquad . \tag{4.17}$$

The function $\eta_2(p,\psi,t,x)$ can also be written in terms of the p.d.f. and c.d.f. of the standard normal distribution as

$$\eta_2(p,\psi,t,x) = (1-p)\psi e^{-\psi t + \psi^2/4} \sqrt{\pi}$$

$$\{[\phi(\sqrt{2}(x - \frac{\psi}{2})) + \phi(\frac{\psi}{\sqrt{2}}) - 1](t - \frac{\psi}{2} + J(t\psi - \frac{\psi^2}{2} - 1))$$

$$+ \phi(\frac{\psi}{\sqrt{2}})(tJ\sqrt{2} - \frac{1}{\sqrt{2}} - J\frac{\psi}{\sqrt{2}}) - \phi(\frac{\psi}{\sqrt{2}} - \sqrt{2}x)(tJ\sqrt{2} - \frac{1}{\sqrt{2}}$$

$$- J(\frac{\psi}{\sqrt{2}} + \sqrt{2}x))\}$$

5. Some Numerical Simulations

We present here results of some simulations, which illustrate numer-
ically the characteristics of the procedure developed in the previous
sections. In order to simulate a random variable Y , having a wear-out
distribution with parameters λ, α and τ , proceed as follows. First,
simulate a random variable U , having a uniform distribution on $(0,1)$.
Then, solve for Y the equation

$$\exp\{-\lambda Y - \lambda^{\alpha}(Y-\tau)_+^{\alpha}\} = U \qquad . \qquad (5.1)$$

Finally, determine $X = \min(Y,\Delta)$.

In the present study we restrict attention to the case of $\alpha = 2$.
Accordingly,

$$Y = \begin{cases} -\dfrac{1}{\lambda}\ln U & , \text{ if } U \geq e^{-\lambda\tau} \\[2ex] \tau - \dfrac{1}{2\lambda} + \dfrac{1}{\lambda}\left(\dfrac{1}{4} - \ln U - \lambda\tau\right)^{1/2} & , \text{ if } U < e^{-\lambda\tau} \end{cases} \qquad (5.2)$$

In the following examples we consider a system with $\lambda^{-1} = 200$ [hr] ,
$\Delta = 225$ [hr] , $\tau = 750$ [hr] , $\alpha = 2$ and the prior probability that
$\{\tau \leq 0\}$ is $p = .2$. For the simulation of the random variable
X_n , $(n=1,2,\ldots)$ we apply a value of $\tau_n = (\tau - t_{n-1})_+$. The values of
the posterior probabilities, p_n , and those of the Bayes estimators,
$\hat{\tau}_n$, are computed adaptively after each stage. In Table 5.1 we present
the results of such a simulation run.

Table 5.1. A Simulation Run with Parameters
$\lambda = 1/200$, $\psi = 1/1000$, $p = .2$, $\tau = 750$

n	X_n	J_n	t_n	p_n	$\hat{\tau}_n$
1	41.49	1	41.	0.288	742.71
2	156.56	1	198.	0.473	646.85
3	108.39	1	306.	0.631	514.63
4	225.00	0	531.	0.437	921.39
5	225.00	0	756.	0.309	1317.66
6	31.19	1	788.	0.385	1237.91
7	23.03	1	811.	0.446	1170.46
8	115.07	1	926.	0.610	1011.46
9	114.04	1	1040.	0.742	866.27
10	20.53	1	1060.	0.779	815.35
11	27.09	1	1087.	0.820	758.42

We see in the above example that the first value of t_n larger than $\hat{\tau}_n$ is $t_n^* = 1040$ [hr] . This is greater than τ by 290 [hr] , which is a little over than 1 MTBF. The corresponding value of p_n is $p_n^* = .742$. The estimate of τ is $\hat{\tau}_n = 866.27$. We see also that if we defer the decision until $p_n \geq .8$ then $t_n^* - \tau = 337$ [hr] and $\hat{\tau}_n = 758.42$. Thus, the second decision (stopping) time provided a more accurate estimate of τ . In Table 5.2 we present frequency distributions and the exact means and standard deviations of $W_n^* = t_n^* - \tau$ as obtained by M=100 independent simulation runs, for each one of the prior parameters $\psi = 1/500, 1/750$ and $1/1000$. The decision time in these simulations is the first value of t_n greater than $\hat{\tau}_n$, i.e., $t_n^* = $ least n≥1, such that $t_n \geq \hat{\tau}_n$
(5.3)

Table 5.2. <u>Frequence Distributions, Means and Standard Deviations of</u> W_n^* <u>in M=100 Simulation Runs</u>

Mid Point Interval	Frequencies		
	$1/\psi=500$	$1/\psi=750$	$1/\psi=1000$
-500	10	0	0
-400	35	28	15
-300	17	37	22
-200	21	7	19
-100	8	11	6
0	0	0	10
100	6	7	10
200	3	9	5
300	0	1	7
400	0	0	6
Mean	-278.15	-218.44	-11?.72
St. Dev.	167.44	201.88	239.82

We see in Table 5.2 that the decision time t_n^* tends to yield too many early decision points. If $1/\psi=1,000$ [hr] the results are significantly better than in the case of $1/\psi=500$ or 750 [hr] .

The situation seems to be better in the case of the decision times defined by

$$t_n^{**} = \text{least } n \text{ s.t. } p_n \geq .9 \qquad\qquad (5.4)$$

In Table 5.3 we present the frequency distributions of $W_n^{**} = t_n^{**} - \tau$, obtained in independent simulation runs.

Table 5.3. Frequency Distributions, Means and Standard
Deviations of W_n^{**} in M=100 Simulation Runs

Mid Point of Intervals	Frequencies		
	$1/\psi=500$	$1/\psi=750$	$1/\psi=1000$
-300	15	2	2
-200	17	22	19
-100	17	12	13
0	10	16	10
100	6	0	4
200	4	7	7
300	8	5	7
400	2	4	3
500	12	17	6
600	9	13	25
700	0	2	4
Mean	65.60	152.85	202.13
St. Dev.	311.78	325.40	322.37

It seems that the Bayes decision times t_n^{**} based on the prior distribution (3.5), with a proper choice of the ψ value (not too small), provide good results in the problem of detecting a shift to the wear-out phase.

References

[1] Mann, N.R., Schafer, R.E. and N.D. Singpurwalla, <u>Methods for Statistical Analysis of Reliability and Life Data</u>, John Wiley and Sons, New York, 1974.

[2] Shriyaev, A.N. (1963). On Optimum methods in quickest detection problems. <u>Theory of Probability and Its Applications</u>, 8: 22-46.

[3] Zacks, S., <u>The Theory of Statistical Inference</u>, John Wiley and Sons, New York, 1971.

REPORT DOCUMENTATION PAGE	READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
1	AD-A117 865	

4. TITLE (and Subtitle)	5. TYPE OF REPORT & PERIOD COVERED
BAYES ADAPTIVE ESTIMATION OF THE POINT OF SHIFT TO THE WEAR-OUT PHASE OF RELIABILITY SYSTEMS	Technical Report
	6. PERFORMING ORG. REPORT NUMBER N00014-81-K-0407

7. AUTHOR(s)	8. CONTRACT OR GRANT NUMBER(s)
S. Zacks	240-6177A

9. PERFORMING ORGANIZATION NAME AND ADDRESS	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
Department of Mathematical Sciences SUNY-Binghamton Binghamton, NY 13901	NR 042-276

11. CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE
	August 2, 1982
	13. NUMBER OF PAGES 18

14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS. (of this report)
Office of Naval Research Arlington, VA 22217	non-classified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Wear-out, Reliability System, Bayes Adaptive Procedure, Stopping Times, Failure Rate Functions, Simulations

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A new family of life distributions, called the wear-out distributions, is developed on the basis of a failure rate function, which is a constant up to the change-point and strictly increasing afterwards. Properties of these wear-out distributions are derived and a Bayes adaptive procedure is developed for the estimation of the change point. Recursive formulae are given for the determination of the posterior probability that the change has occurred and of its Bayes estimator. The results of numerical simulations are given to illustrate the properties of the adaptive procedure.

DD FORM 1473 EDITION OF 1 NOV 65 IS OBSOLETE
1 JAN 73
S/N 0102-LF-014-6601